Inverse Model for Solving Partial Differential Equations Using a Stochastic Method and COMSOL Multiphysics®

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Abstract: Partial differential equations (PDEs) could explain the theory of many phenomena in nature. Since these equations are complicated to be solved theoretically except for simple structures with simplifying assumptions, numerical method are mainly employed to solve PDEs. Also, there are many applications described by PDEs with multiple inputs and outputs which are generally optimization problems. Microwave imaging with several transmitting and receiving antennas is one of such problems, which needs large number of PDEs to be solved, and as a result large amount of time and memory is needed. In order to reduce the cost of solution, the main problem was transformed to a corresponding stochastic optimization problem, and defined cost function is calculated for different optimization parameters.

Keywords: partial differential equations, stochastic optimization, microwave imaging, breast cancer

1. Introduction

Partial differential equations (PDEs) explain many phenomena in the nature. They can describe fluid flow, thermal behavior, mechanic structures, microwave and many more. These equations can be simplified and solved theoretically for simple problems. But, for complicated structures, which exist in the real world, it is not possible to find the answer simply. In order to overcome this issue, numerical methods are used, and simulation software like COMSOL Multiphysics has been developed.

In many cases it is needed to solve inverse partial differential equations. Solving these problems involves minimizing the defined error function in a loop. In each step PDE forward problem is solved and desired parameters are adjusted accordingly to meet the criterion. If the number of PDEs needed to be solved in each step is increased, computational time and memory requirement are increased. In order to speed up the solution process, Haber et al. proposed a method to reformat the problem to a stochastic optimization problem [1]. In this method only one PDE is needed to be solved, so the cost of solution can be reduced.

In this paper, we used the LiveLink for MATLAB interface with COMSOL Multiphysics to calculate the cost function for different radii of a scatterer in a microwave problem with multiple input and output antennas. COMSOL was used to solve forward problem in our example. The LiveLink features allowed us to generate a function which create outputs based on desired inputs in each step of optimization loop. Using this function, allows to include the finite element solution of the problem in the whole process of solving the inverse problem.

2. PDEs Describing Microwave Imaging [2]

Microwave imaging is governed by Maxwell's equations which are set of partial differential equations. Since microwave antennas operate in a single frequency, these equations can be written in pharos format:

$$\begin{aligned} \nabla \times E &= -j\omega\mu H \\ \nabla \times H &= J + j\omega\varepsilon E \\ \nabla \cdot E &= \rho / \varepsilon \\ \nabla \cdot H &= 0 \end{aligned} \tag{1}$$

where E is electric field intensity, H is magnetic field intensity, and ω, J, ε, μ and ρ are angular frequency, current volume density, medium’s permittivity and permeability and volume charge density respectively.

In a source-free medium where $\rho = 0$ wave propagation can be represented as:

$$\nabla^2 E + k^2 E = 0$$
$$\nabla^2 H + k^2 H = 0 \tag{2}$$

in these equations k is wave number and is expressed as follows:

$$k = \omega\sqrt{\mu\varepsilon} \tag{3}$$

Boundary conditions applied to this problems are:

$$\hat{n} \times (E_1 - E_2) = 0$$
$$\hat{n} \times (H_1 - H_2) = J_s$$
$$\hat{n} \cdot (D_1 - D_2) = \rho_s \tag{4}$$
$$\hat{n} \cdot (B_1 - B_2) = 0$$

3. Stochastic Optimization

There are several approaches to solve PDE inverse problems. But, all of them are common in minimization of cost function in a loop until it meets desired criteria. Since the forward problem is needed to be solved in each loop, according to the size of this problem, the process could be very expensive regarding time and memory.

Many problems of this type consists of PDEs with multiple right hand side, which dramatically increases the cost of solving. So, Haber et al. suggested to convert such problems to a stochastic optimization problem which is much cheaper to solve [1].

Assume that we want to solve following problem:

$$\min \left\| F(x)I - M \right\|^2 \tag{5}$$

In order to reduce required solving time and memory, this problem can be transformed to a stochastic optimization problem by multiplying it with a random variable:

$$\min \left\| (F(x)I - M)\omega \right\|^2 \tag{6}$$

As a result, the new problem needs to solve only one forward function, which is a PDE function in this case, on each step, and consequently helps to reduce computational cost significantly.

The new generated stochastic optimization problem which corresponds to the main optimization problem can be solved using optimization algorithms for stochastic problems such as Stochastic Approximation and Stochastic Average Approximation [3].

4. COMSOL Simulation

Microwave imaging reconstruction is one of the very expensive optimization problems because it involves solving multiple PDEs in each step. And, the solution cost increases when several transmitting and receiving antennas are used to get better results.

We modeled an example system of a microwave imaging for breast cancer detection which is shown in Figure 1. In this model breast is immersed in a tank of coupling liquid to reduce the reflected incident power. In this example skin and chest wall is modeled and breast tissue is considered to be homogenous and only one tumor which is generally called scatterer in this paper is implemented in the model.

This model is solved in the frequency of 2GHz, and properties of different materials used in this example are illustrated in Table 1 [4]. Electromagnetic field distribution is calculated for two different transmitting antennas. Figures 2 and 3 clearly show the scatterer's effect on electromagnetic field when different antennas are excited.

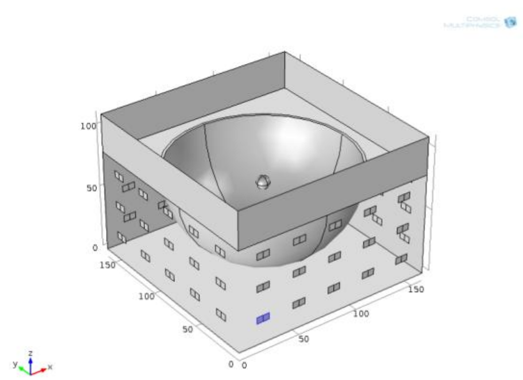

Figure 1. Microwave imaging with multiple transmitting and receiving antennas

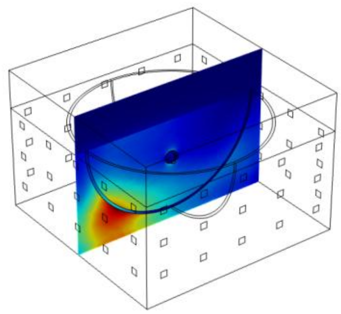

Figure 2. Scatterer's effect on microwave field

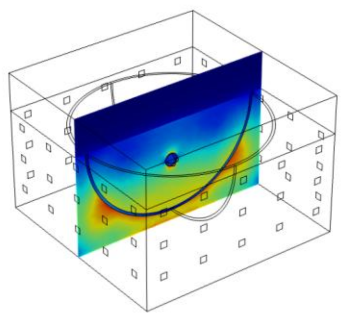

Figure 3. Scatterer's effect on microwave field

It is reasonable to use stochastic optimization method described in the previous section for this structure. Because it consists of multiple input and output antennas, and solving the inverse problem implies solving several PDEs at each optimization step, which can be reduced by using stochastic approach.

4. Results

Here, as a first step of implementation of the described stochastic optimization algorithm, we use the simplified version shown in Figure 4. In this version, skin is not modeled and the whole tank is considered to be filled with the medium with the properties of the homogeneous breast tissue.

We directly use COMSOL as a forward model in optimization process. MATLAB link of the COMSOL allows us to create the function with desired optimization parameters that can be easily used in optimization loops. We applied stochastic average approximation method to this model to calculate the values of cost function for different values of scatterer's radii. The problem is solved for five realization of the random variable, and final result is calculated by averaging these outputs. The result is shown in Figure 5.

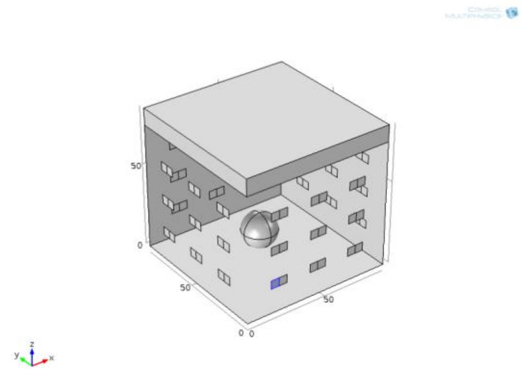

Figure 4. Simplified model

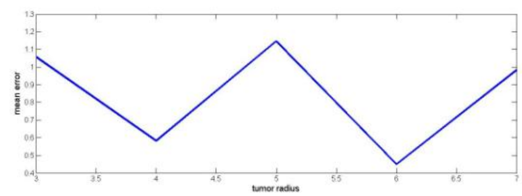

Figure 5. Cost function value for different scatterer's radius

Table 1: Tissue Properties

Tissue	Relative permittivity	Conductivity (S/m)
Immersion liquid	9	0
Chest wall	50	7
Skin	36	4
Breast tissue	9	0.4
Tumor	50	4

5. Conclusion

COMSOL gives an excellent tool to use finite element in optimization process by using LiveLink for MATLAB module. This helps to use more accurate forward model in this process which considers more details and have results much closer to the real world. In this paper we use it to solve a microwave imaging optimization process. Since we have a large number of inputs and outputs, we use the stochastic optimization method long with FEM forward model for a simplified breast cancer detection system to calculate values of a defined cost function.

8. References

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