Dimensionless versus Dimensional Analysis in CFD and Heat Transfer

Heather E Dillon*[,1], Ashley Emery[1], RJ Cochran[2], and Ann Mescher[1]
[1]University of Washington
[2]Applied CHT, Seattle, WA
*Corresponding author: Seattle, WA 98105, hedillon@u.washington.edu

Abstract: Students in engineering and science are often exposed early in their studies to non dimensional analysis. This is particularly true in the areas of fluid mechanics and heat transfer where most experimental correlations are expressed in terms of non dimensional groups and many numerical analysis involve the solution of non dimensional equations. When it comes to solving fluid flow/heat transfer problems, many solutions, particularly industrial ones, are based on finite element/finite volume programs that use dimensioned quantities. In order to compare to reference information one would like to use codes like COMSOL Multiphysics (basically a dimensional code) to solve non dimensional problems. Unfortunately there are many ways of non dimensionalizing the problem and it is not clear which is best. For example in convection a reference velocity can be expressed as $\nu/L\sqrt{Gr}$, $\nu/L\sqrt{Ra}$, $\nu/L\sqrt{Re}$, ν/L, or α/L where ν and α are the kinematic viscosity and thermal diffusivity and L is some length scale and Re, Gr and Ra are the Reynolds, Grashof and Rayleigh numbers. The different choices lead to different solution techniques, particularly for highly non linear problems and to different interpretations. Some non dimensionalizations are more appropriate than others. The paper describes numerical experiments computing a highly transient free convection flow, compares dimensionless and dimensional results, the solution techniques used, and discusses which and why specific choices of non dimensional groups are more appropriate to different flow situations.

Keywords: dimensionless, heat transfer, comsol, natural convection

1 Introduction

Using a dimensioned code like COMSOL to compute non-dimensional solutions may be useful for many reasons. One key reason might be to compare to a benchmark solution or to calibrate with a key dimensionless parameter.

To illustrate one method of comparison an example problem for natural convection in a tall cavity has been developed with a known dimensionless solution. The benchmark solution is then compared to a traditional CFD approach that might be contemplated by many users of a tool like COMSOL where a dimensioned solution is considered. The boundary conditions for the example problem are shown in Figure 1.

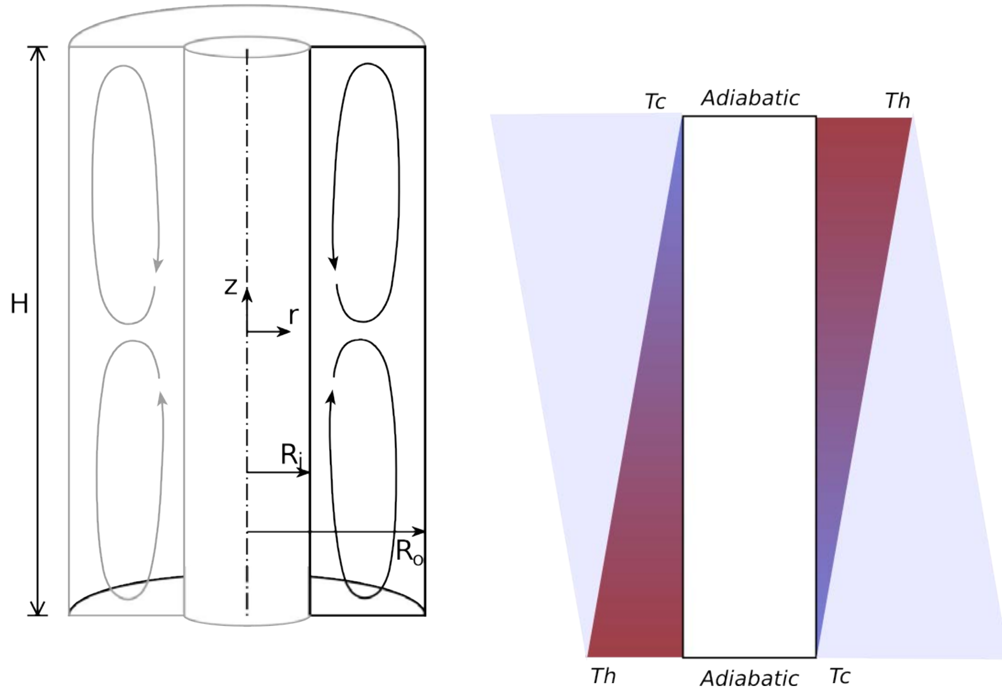
Several dimensionless parameters are used to characterize the flow. The Rayleigh number (Ra) is defined below and is often used in buoyancy flows to characterize the transition between conduction dominated flow and convection dominated flow. The Prandlt number (Pr) is the ratio of the viscous diffusion and thermal diffusion. For this work, only the value of $Pr = 0.71$ (representative of air) is considered and $Ra = 2.5e7$.

$$Ra = \frac{\rho^2 g c_p \beta(\Delta T) L^3}{k\mu} \tag{1}$$

c_p is the heat capacity of the gas, g is the acceleration of gravity, β is the isobaric coefficient of thermal expansion, μ is the dynamic viscosity, k is the thermal conductivity, and ρ is the density.

$$Pr = \frac{c_p\mu}{k} \tag{2}$$

The critical Rayleigh number (Ra_c) is an important value for comparisons to experimental work. It is defined as the Rayleigh number at which the flow transitions from one stability regime to another, corresponding to the transitions from the conduction



(a) Schematic of the computational domain. (b) Boundary conditions.

Fig. 1: Linear temperature profile applied to the vertical walls of the annulus.

dominated regime to convection dominated regime. In this example case problem the critical Rayleigh number characterizes the formation of cells with oscillatory temperatures. The heat transfer across the cavity is usually reported in terms of the Nusselt number (Nu), which is the ratio of the convective heat transfer coefficient to the conduction heat transfer coefficient.

Dimensionless analysis in natural convection is often further complicated by the use of the Boussinesq approximation that is used to further simplify the governing equations for the system. This approximation is also considered in the results presented and is discussed in more detail by Tritton [18]. Table 1 gives an overview of the common dimensionless numbers used in natural convection flows.

Parameter	**Equation**	**Description**
Grashof	$Gr = \frac{g\beta(\Delta T)L^3}{\nu^2}$	Ratio of buoyancy forces to viscous forces.
Nusselt	$Nu = hL/k$	Ratio of convection to conduction heat transfer in the fluid.
Peclet	$Pe = RePr$	Ratio of the rate of advection to the rate of diffusion in the fluid.
Prandtl	$Pr = c_p\mu/k$	Ratio of the momentum and thermal diffusivities.
Rayleigh	$Ra = \frac{g\beta(\Delta T)L^3}{\nu\alpha}$	The product of Pr and Gr. Fluid specific ratio of buoyancy forces to viscous forces.
Reynold	$Re = \frac{\rho VL}{\mu}$	Ratio of inertial forces to viscous forces.

Table 1: Summary of important dimensionless parameters in natural convection.

Author	Year	A	Description
De Vahl Davis [5]	1983	1	Benchmark solution for a square cavity.
Lee and Korpela [9]	1983	0-1000	Reported Nu and streamfunctions.
Chenoweth and Paolucci [2]	1986	1-10	Compared ideal gas and Boussinesq.
Suslov and Paolucci [17]	1995	∞	Non-Boussinesq impact on stability, considered Ra_c with ΔT.
Mlaouah et al. [11]	1997	1	Compared Boussinesq, ideal gas, and low Mach approximation.
Paillere et a. [13]	2000	1	Compared Boussinesq and low Mach approximation.
Xin and Le Quere [20]	2002	8	Benchmark study reported Ra_c.
Christon et al. [3]	2002	8	Comparison study of methods, grids, etc.
Reeve et al. [16]	2003	10	Commercial code FIDAP.
Vierendeels et al. [19]	2003	1	Benchmark with ideal gas.
Xin and Le Quere [21]	2006	1-7	Investigated instabilities.
Dillon et al. [6]	2009	8-33	Dimensioned benchmark study in rectangular cavity, COMSOL.

Table 2: Summary of experimental and computational natural convection studies in a tall cavity (annular and rectangular geometries) for $Pr = 0.71$.

For natural convection studies in a cavity several parameters are used to characterize the systems. The geometry of the cavity is represented by the height H, the width L, the aspect ratio (A) and the radius ratio (η). For the computational work described in this paper the aspect ratio considered was $A = 10$ and the radius ratio was $\eta = 0.6$. R_o and R_i are the outer and the inner radius of the annulus, respectively.

$$A = \frac{H}{L} \quad (3)$$

$$\eta = \frac{Ro}{R_i} \quad (4)$$

2 Literature

The literature has shown that the stability of the flow in a cavity is governed by the Prandtl number, the Rayleigh number and the geometry of the cavity. At small Rayleigh numbers the flow is dominated by conduction. As the Rayleigh number is increased the flow becomes unstable, first resulting in multicellular secondary flow patterns, and then as the Rayleigh number is further increased the flow becomes chaotic.

The focus in most experimental and computational work is on small aspect ratio cavities ($A = 1 - 5$) including de Vahl Davis [5]. For small aspect ratios the flow is not multicellular and is time invariant.

The computational models for this problem are summarized in Table 2. All the computational studies, unless otherwise noted, used the Boussinesq approximation. As early as 1986 authors considered the impact of the Boussinesq approximation and compared it to a full ideal gas simulation ([2] and [17]). Mlaouah et al. [11] also compared the Boussinesq approximation to the low-Mach approximation.

Some authors ([14] and [7]) predict an oscillatory time-dependent instability for tall cavities. Lee and Korpela [9] and Liakopoulos [10] predict that a stationary instability precedes the onset of oscillatory convection for high aspect ratios. Suslov and Paolucci indicate that this observation is simply a product of temperature invariant air properties [17]. They also observe that two modes of instability are possible depending on the magnitude of the temperature difference between the hot and cold wall.

In a similar work Chenoweth and Paolucci [2] observed that as the temperature difference in the cavity increases, a lower critical Rayleigh number is found. They confirmed that the nature of the instability changes with increasing temperature difference.

Many of these results imply that a purely dimensionless approach to natural convection does not fully capture the temperature dependent nature of the air properties which may impact the computed natural convection behavior. However many of the key benchmarks in the field use dimensionless simulations and the Boussinesq approximation, so performing comparison simulations may be of interest to many researchers.

3 Solution Methods

Two specific types of problems were considered for this paper, a traditional dimensionless simulation and a practical dimensioned simulation. Both solutions were developed using the COMSOL Multiphysics package 3.5a.

3.1 Governing Equations

The buoyancy driven flow is modeled as a coupled system with fluid motion (Navier Stokes), convection and conduction heat transfer. The governing equations are given in Equations 5-8.

The boundary conditions for the system are specified as adiabatic on the upper and lower cavity boundaries. The right and left walls are constrained as varying linearly in the z-direction.

The weakly compressible Navier Stokes equation for this system is given in simplified form.

$$\rho\frac{\partial u}{\partial t} + \rho(u \cdot \bigtriangledown)u = \bigtriangledown \cdot (-pI + \eta(\bigtriangledown u + \bigtriangledown u^T)) - (2\eta/3 \bigtriangledown \cdot u)I + F \tag{5}$$

$$\bigtriangledown \cdot u = 0 \tag{6}$$

$$\bigtriangledown \cdot (c \cdot \psi) = 0 \tag{7}$$

In this form u represents the velocity vector of the fluid, ρ is the fluid density, η is the dynamic viscosity of the fluid, and F is the force applied to the fluid, c is the diffusion coefficient and ψ is the streamfunction. The force term is varied depending on which type of analysis is applied.

The heat transfer convection is governed by Equation 8.

$$\rho c_p \frac{\partial T}{\partial t} + \bigtriangledown \cdot (-k \bigtriangledown T) = -\rho c_p u \cdot \bigtriangledown T \tag{8}$$

For this equation k is the thermal conductivity, c_p is the heat capacity at constant pressure, and T is the temperature.

3.2 Air Properties

The effect of the variable density in natural convection systems is often represented by the Boussinesq approximation. The Boussinesq approximation is used for computational problems of this type to simplify the formation of the coupled Navier-stokes equations. The approximation assumes that density variations are small in the fluid except in evaluating the buoyancy force (gravity multiplied by density). In general, the Boussinesq approximation is only valid when the temperature differences in the system are small (less than 28^oC for air).

For the system considered in this work the Boussinesq approximation is valid, however some authors did observe different results when the air was treated as an ideal gas ([2] and [17]). For this reason the Boussinesq approximation was used for the dimensionless analysis and compared to treating the air as an ideal gas in the dimensioned analysis.

Other fluid properties (k, c_p, μ) were all temperature independent as defined by Reeve [15].

3.3 Spatial Discretization

For this analysis a grid resolution study was conducted to determine the required mesh density and to compare the results with those of Reeve [16] using FIDAP. The mesh refinement study did not include the time step parameter because in COMSOL the best results are achieved using a dynamic time step calculated by the software based on the CFL number.

Author	Characteristic Velocity	Description
	$\frac{\alpha}{L} = \frac{k}{\rho c_p L}$	Thermal diffusion velocity.
De Vahl Davis [5]	$\frac{\sqrt{\beta g \Delta T L}}{\sqrt{Gr}} = \frac{\mu}{\rho L}$	Viscous diffusion velocity.
Ostrach [12]	$\sqrt{\beta g \Delta T L}$	For strongly coupled flows $Pr < 1$ and $\sqrt{Gr} > 1$
Ostrach [12]	$\frac{\sqrt{\beta g \Delta T L}}{\sqrt{Pr}}$	For strongly coupled flows $Pr > 1$ and $\sqrt{Gr} > 1$
Wan Hassan [8]	$\frac{\alpha Ra^{1/4}}{L} = \frac{k Ra^{1/4}}{\rho c_p L}$	Based on boundary layer thickness and thermal diffusion velocity.
Abrous [1]	$\frac{\mu Ra^{1/4}}{\rho L}$	Based on boundary layer thickness and viscous diffusion velocity.

Table 3: Summary of characteristic velocity for natural convection systems. Adapted from Cochran [4].

The rectangular mesh system used for the COMSOL results is the same as the mesh used by Reeve. The triangular mesh type denotes the dynamic free meshing system built into COMSOL. In general, the free mesh is more robust for natural convection problems because COMSOL automatically refines the mesh in the corners of the domain to capture boundary layer effects. Agreement between FIDAP and COMSOL is good for the rectangular meshes. Results for the 1120 element triangular mesh were also consistent with the results of Reeve and this triangular mesh was adopted for future computations because the convergence of the system was more robust.

4 Dimensionless Options

Many options are available for creating a dimensionless system in natural convection. The way a non-dimensional system is defined should not change the total result in the system. However, depending on the way the system is scaled numerical rounding may influence the accuracy of the result. Although each result will scale between 0 and 1 the precision of the values may change the results slightly based on the computational tool and number of significant figures carried in each calculation.

The definition of the characteristic velocity is a key element in the dimensionless system with options shown in Table 3. For weakly coupled systems the characteristic velocity is usually based on the forced convection in the system. For strongly coupled (natural convection or buoyancy driven) flows Ostrach [12] recommends different forms of the characteristic velocity depending on the Prandtl number. For this work only the first form is of interest because all simulations are for $Pr = 0.71$. Other important characteristic velocities are listed including the viscous diffusion velocity used by De Vahl Davis [5] and the thermal diffusion velocity. Wan Hassan and Abrous use variations of these forms that are similar for gases but may diverge significantly from one another for other fluids.

Three options are outlined in Table 4. For consistency with prior work this system was modeled using dimensionless parameters and the Boussinesq approximation. The height of the cavity H is the characteristic length and the temperature difference between the hot and cold wall is the characteristic temperature, $\Delta T = (T_h - T_c)$. Based on these characteristics the dimensionless spatial coordinates (Z, R), velocities (U, V), time (τ), temperature (Θ) and pressure (P) are given in Table 4 as they would need to be implemented in a tool like COMSOL. The COMSOL documentation provides instructions only for implementing the Boussinesq approximation for dimensioned flows.

Parameter	Option 1	Option 2 Strongly coupled	Option 3 Weakly coupled
R, Z	$\frac{r}{H}, \frac{z}{H}$	$\frac{r}{H}, \frac{z}{H}$	$\frac{r}{H}, \frac{z}{H}$
U	$\frac{u}{\frac{\alpha}{L}\sqrt{RaPr}}$	$\frac{u}{\sqrt{g\beta\Delta TH}}$	$\frac{u}{u_{forced}}$
V	$\frac{v}{\frac{\alpha}{L}\sqrt{RaPr}}$	$\frac{v}{\sqrt{g\beta\Delta TH}}$	$\frac{v}{u_{forced}}$
Θ	$\frac{T-T_c}{\Delta T}$	$\frac{T-T_c}{\Delta T}$	$\frac{T-T_c}{\Delta T}$
τ	$t\sqrt{g\beta\Delta T/H}$		
P	$\frac{pL}{\mu\frac{\alpha}{L}\sqrt{RaPr}}$		
ρ	$\sqrt{\frac{Ra}{Pr}}$	1	1
c_p	Pr	1	1
μ	1	$\sqrt{\frac{Pr}{Ra}}$	$\frac{1}{Re}$
g	1		
β	1	1	NA
k	1	$\frac{1}{\sqrt{RaPr}}$	$\frac{1}{Pe}$
F	$(T-T_c)\sqrt{\frac{Ra}{Pr}}$	$\sqrt{\frac{Ra}{Pr}}$	Re

Table 4: Summary of dimensionless parameter implementation in a dimensioned tool.

5 Results

The simulation of natural convection in a tall annulus at $Ra = 2.5e7$ results in a steady, oscillatory behavior. Figures 1- 3 give an overview of the streamfunction and temperature contours for the simulation solution for Option 1, dimensionless analysis. The period and amplitude of the temperature oscillations in the center of the cavity are computed to compare the dimensioned and dimensionless options.

The results for the simulation when performed with Option 1 of the simulation techniques discussed above are shown in Table 5. Option 3 is not considered because the sample simulation is only for natural convection (strongly coupled) flows. The dimensionless version compares well with the dimensioned version in COMSOL. The dimensioned simulation was run with very low temperature differences (1K) to reduce any error that might result from linearization of the density in the Boussinesq approximation. The small errors that result in the dimensioned solution when compared to the dimensionless solution for the Boussinesq version are due in part to rounding error in the calculation of the Rayleigh number and values of the air constants.

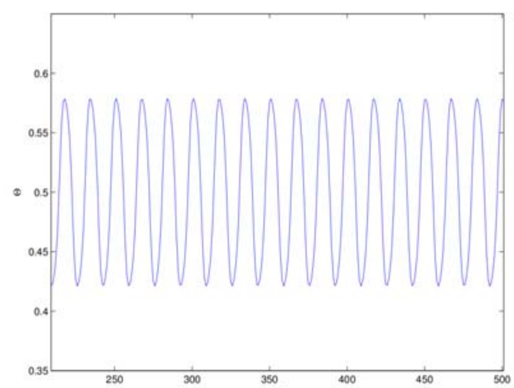


Figure 1: Dimensionless temperature at the center of the cavity over time. $Ra = 2.5e7$, $A = 10$ and $\eta = 0.6$.

As expected, the ideal gas solution is very close to the Boussinesq solution for the small temperature difference considered (less than 1K) however the critical Rayleigh number and oscillation shape may change as the temperature difference is increased. When temperature variations in the conductivity, heat capacity and viscosity were simulated the amplitude of the oscillation changed more than did the frequency. This is consistent with the expectation that the air properties may alter the buoyancy effect on the hot and cold sides as the temperature variations in the field adjust the properties.

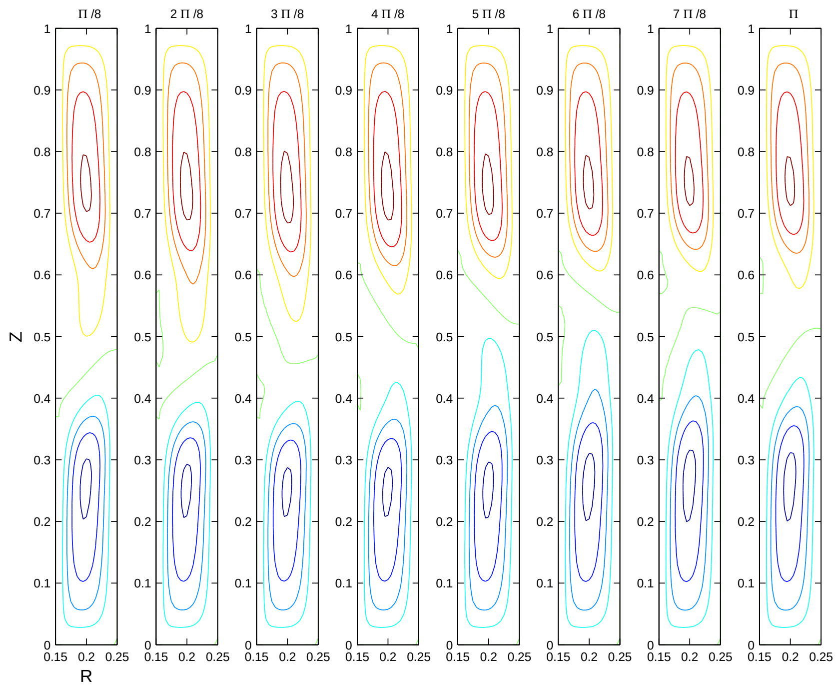


Fig. 2: Sequential contour plot of the stream function illustrating oscillation of the natural convection cells through one period (Π). $Ra = 2.5e7$, $A = 10$ and $\eta = 0.6$.

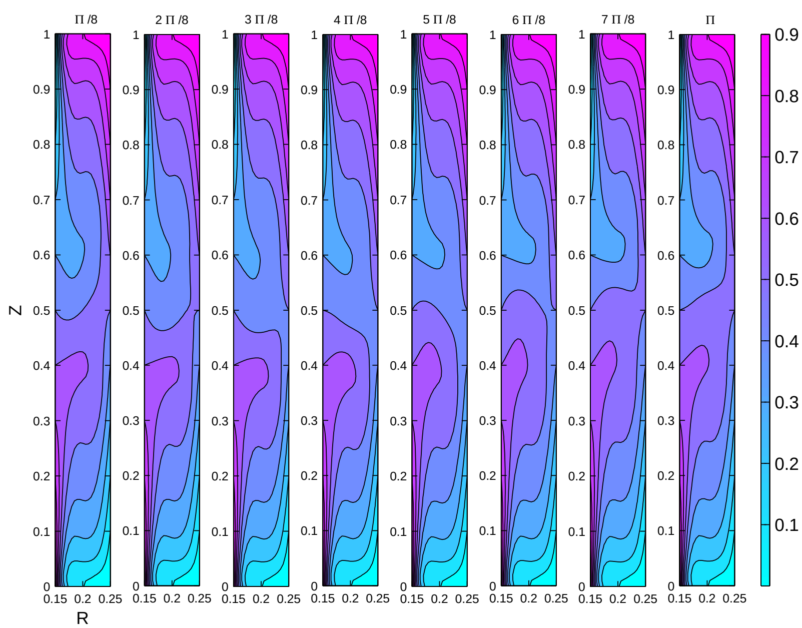


Fig. 3: Sequential contour plot of the temperature (Θ) illustrating oscillation of the natural convection cells through one period (Π). $Ra = 2.5e7$, $A = 10$ and $\eta = 0.6$.

Description	Rayleigh	Density	k, cp, μ	Period	Amplitude
Option 1 [16]	2.5e7	Boussinesq	Constant	16.15	0.1285
Option 1	2.5e7	Boussinesq	Constant	16.126	0.1278
Dimensioned	2.5e7	Boussinesq	Constant	16.116	0.1279
Dimensioned	2.5e7	Ideal Gas	Constant	16.161	0.1277
Dimensioned	2.5e7	Ideal Gas	T dependent	16.161	0.1253
Option 1	4e7	Boussinesq	Constant	11.0703	0.1484
Dimensioned	4e7	Boussinesq	Constant	11.0091	0.1485
Dimensioned	4e7	Ideal Gas	Constant	11.0091	0.1485
Dimensioned	4e7	Ideal Gas	T dependent	11.0091	0.1481
Option 1	10e7	Boussinesq	Constant	7.1860	0.1361
Dimensioned	10e7	Boussinesq	Constant	7.0917	0.1362

Table 5: Comparison of simulation options in COMSOL for $A = 10$ and $\eta = 0.6$ in the center of the air cavity. Each simulation was performed with the 1120 element triangular mesh except the results of Reeve [16]. The dimensioned results have been converted to non-dimensional units based on the definitions in Table 4

Temperature dependent results for $Ra = 10e7$ are not presented because the simulations showed the flow to be significantly different in character from the constant property solutions. In general, simulations using temperature dependent properties follow a different solution path as the Rayleigh number is increased. This is a function of the intrinsic chaotic behavior of the system. This phenomena has also been observed at higher Rayleigh numbers when the Boussinesq approximation is used [16]. The complexities of the chaotic behavior are documented in a future publication.

6 Conclusions

The results for natural convection in a tall cavity with linear boundary conditions are reported and agree well with the results of Reeve [16]. Oscillatory flow is predicted with a triangular free mesh with 1120 elements that agrees well with the rectangular mesh used in prior work. The natural convection flow creates a multi-cellular flow pattern which creates regular temperature oscillations in the air cavity.

Using the commercial tool COMSOL it has been shown that for a cavity with linear temperature boundary conditions the dimensioned and dimensionless solutions agree well using the Boussinesq approximation. Using the ideal gas equation has a very small impact as long as the temperature difference is small but near the Boussinesq limit simulations should use ideal gas rather than Boussinesq since the computational time increase is very small in modern systems.

In all the cases considered the variation of other air properties (k, cp, μ) with temperature had a larger impact on the solution than other modeling decisions. This result is consistent with prior authors ([17]) who found that the behavior of the oscillatory phenomena in cases like the one considered is dependent on variation of air properties with temperature.

Acknowledgments

These results were obtained as part of the research supported by the National Science Foundation through Grant 0626533.

References

[1] A. Abrous, *A numerical study of the turbulent free convection in rectangular cavities*, Ph.D. thesis, University of Washington, 1988.

[2] D.R. Chenoweth and S. Paolucci, *Natural convection in an enclosed vertical air layer with large horizontal temperature differences*, Journal of Fluid Mechanics **169** (1996), 173–210.

[3] M.A. Christon, P.M. Gresho, and S.B. Sutton, *Computational predictability of*

time-dependent natural convection flows in enclosures (including a benchmark solution), International Journal for Numerical Methods in Fluids **40** (2002), 953–980.

[4] Robert J. Cochran, *Laminar and turbulent incompressible fluid flow analysis with heat transfer by the finite element method*, Ph.D. thesis, University of Washington, 1992.

[5] G. de Vahl Davis, *Natural convection of air in a square cavity: A bench mark numerical solution*, International Journal for Numerical Methods in Fluids **3** (1983), 249–264.

[6] H.E. Dillon, A.F. Emery, and A.M. Mescher, *Benchmark comparison of natural convection in a tall cavity*, COMSOL User Conference, 2009.

[7] P. Haldenwang and G. Labrosse, *2-d and 3-d spectral chebysheve solutions for free convection at high rayleigh number*, Sixth International Symposium on Finite Element Methods in Flow Problems (1986).

[8] M. Wan Hassan, *The natural convection in a cavity at high rayleigh numbers*, Ph.D. thesis, University of Washington, 1986.

[9] Y. Lee and S. A. Korpela, *Multicellular natural convection in a vertical slot*, Journal of Fluid Mechanics **126** (1983), 91–121.

[10] A. Liakopoulos, P.A. Blythe, and P.G. Simpkins, *Convective flows in tall cavities*, Simulation and Numerical Methods in Heat Transfer (1990), 81–87.

[11] H. Mlaouah, T. Tsuji, and Y. Nagano, *A study of non-boussinesq effect on transition of thermally induced flow in a square cavity*, International Journal of Heat and Fluid Flow **18** (1997), no. 1, 100–106.

[12] S. Ostrach, *Natural convection in enclosures*, J. Heat Transfer **110** (1988), no. 4, 1175–1190.

[13] H. Paillere, C. Viozat, A. Kumbaro, and I. Toumi, *Comparison of low mach number models for natural convection problems*, Heat and Mass Transfer **36** (2000), 567–573.

[14] S. Paolucci and D.R. Chenoweth, *Transition to chaos in a differentially heated vertical cavity*, Journal of Fluid Mechanics **201** (1989), 379–410.

[15] Hayden Reeve, *Effect of natural convection heat transfer during polymer optical fiber drawing*, Ph.D. thesis, University of Washington, 2003.

[16] Hayden M. Reeve, Ann M. Mescher, and Ashley F. Emery, *Unsteady natural convection of air in a tall axisymmetric, nonisothermal annulus*, Numerical Heat Transfer, Part A **45** (2004), 625–648.

[17] S.A. Suslov and S. Paolucci, *Stability of natural convection flow in a tall vertical enclosure under non-boussinesq conditions*, International Journal of Heat and Mass Transfer **38** (1995), no. 12, 2143–2157.

[18] D.J. Tritton, *Physical fluid dynamics*, Oxford Science Publications, 1988.

[19] J. Vierendeels, B. Merci, and E. Dick, *Benchmark solutions for the natural convective heat transfer problem in a square cavity with large horizontal temperature differences*, International Journal of Numerical Methods for Heat and Fluid Flow **13** (2003), no. 8, 1057–1078.

[20] S. Xin and P. Le Quere, *An extended chebyshev pseudo-spectral benchmark for the 8:1 differentially heated cavity*, International Journal for Numerical Methods in Fluids **40** (2002), 981–998.

[21] S. Xin and P. Le Qur, *Natural-convection flows in air-filled, differentially heated cavities with adiabatic horizontal walls*, Numerical Heat Transfer, Part A: Applications **50** (2006), 437–466.